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Canonical Cartan equations for higher order variational problems

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The first problem of global variational Calculus is to try to formulate intrinsically the Euler-Lagrange equations which characterize the critical sections. For variational problems of arbitrary order in one variable and for variational problems in n variables of order 1 or 2 this is attained by means of the Poincaré-Cartan form (see [3], [5], [7] and [8]). It is certainly well-known that in such cases it is possible to associate to each r-order variational problem on a submersion $p : Y \to X$ an ordinary n-form Θ on J^{2r-1} such that the critical sections of p are characterized by the Cartan equation:

$$(*) \qquad (i_D \, \mathrm{d}\Theta)\big|_{j^{2r-1}s} = 0 \quad \text{for every vector field } D \text{ in } J^{2r-1}.$$

Several authors ([1], [2], [4] and [6]) have recently proved, through different methods, that for r-order variational problems in n variables with $r > 2$ and $n > 1$ the Poincaré-Cartan form is not unique and it essentially depends on a linear connection on the base X and on a linear connection on the vertical bundle $V(Y)$. Briefly, the fundamental result of this theory can be summarized in the following way:

«Let $p : Y \to X$ be a submersion of differentiable manifolds, ω a volume element on X and let $\mathscr{L} : J^r \to \mathbb{R}$ be a differentiable function. For each pair of linear connections ∇_0, ∇ on $T(X)$, $V(Y)$, respectively, it is possible to associate to the Lagrangian density $\mathscr{L}\omega$ an ordinary n-form Θ on J^{2r-1} such that the critical sections of the variational problem defined by $\mathscr{L}\omega$ are characterized by the (*) condition.

Globally, the Θ form can be expressed as

$$\Theta = \mathscr{L}\omega + \eta \,\overline{\wedge}\, \theta^r,$$

where η a section of the vector bundle $\Lambda^{n-1} T^*(X) \otimes_{J^{2r-1}} V^*(J^{r-1})$ and θ^r stands for the structure form on J^r.

Similarly the differential of Θ can be expressed as

$$\mathrm{d}\Theta = (\mathrm{d}\mathscr{L}\overline{\wedge}\,\omega + \bar{\mathrm{d}}\Theta) + (\bar{\eta}\,\overline{\wedge}\,\theta^{r})\,\overline{\wedge}\,\theta^{2r-1},$$

where $\bar{\mathrm{d}}$ is the formal differential on J^{∞} (precise definitions will be given afterwards). In this decomposition the $(n+1)$-form $\Phi = \mathrm{d}\mathscr{L}\wedge\omega + \bar{\mathrm{d}}\Theta$ does not depend on the connections chosen and gives the Euler-Lagrange equations, while the second term does depend on such connections and does not appear in the Euler-Lagrange equations (because it contains double products of structure forms) but it determines the pre-symplectic structure on the space of critical sections».

For a more detailed exposition of the development of this methodology one may consult [4].

In this note I shall prove that it is really possible to describe the Φ form by means of simple axioms. More precisely the result is the following.

THEOREM. *Let $p: Y \to X$ be a submersion on a manifold X oriented by a volume element ω and $\mathscr{L}: J^r \to \mathbb{R}$ be a differentiable function. There exists a unique ordinary $(n+1)$ form $\Phi_{\mathscr{L}}$ on J^{2r} which fulfills the following conditions:*

(1) $i_D\Phi_{\mathscr{L}} = 0$ *for every vector field D of $T(J^{2r})$ vertical on Y.*

(2) $i_{D_1}i_{D_2}\Phi_{\mathscr{L}} = 0$ *for every pair of vector fields D_1, D_2 of $V(J^{2r})$.*

(3) *There exists a section η of $\Lambda^{n-1}T^*(X)\otimes_{J^{2r-1}}V^*(J^{r-1})$ such that*

$$\Phi_{\mathscr{L}} = \mathrm{d}\mathscr{L}\wedge\omega + \bar{\mathrm{d}}(\eta\,\overline{\wedge}\,\theta^{r}).$$

A section s of p critical for the variational problem defined by $\mathscr{L}\omega$ if and only if for every vector field D on J^{2r} one has:

$$(**) \qquad (i_D\Phi_{\mathscr{L}})|_{j^{2r}s} = 0.$$

In this way it is always possible to associate to each variational problem an $(n+1)$-form Φ (which replaces $\mathrm{d}\Theta$ but is not exact!) independent of every connection which characterizes the critical sections by a condition of the (*) type.

Vinogradov has obtained a characterization of Euler-Lagrange equations as a differential of a certain spectral sequence. The context of his theory differs from ours and uses different and more sophisticated methods (see for instance [9]).

Before passing on to the proof of this theorem, we shall introduce some notations and known results which will be used later.

a) Given a sumbersion $p: Y \to X$, the k-jet bundle of local sections of p is denoted by $J^k = J^k(Y/X)$, with canonical projections $p_k: J^k \to Y$, $\bar{p}_k: J^k \to X$, and the vertical bundle on X is denoted by $V(J^k)$.

The *vertical differential of order* k of a section s at a point $x \in X$ is the linear map

$$(d_k^v s)_x : T_{j_x^{k-1}s}(J^{k-1}) \to V_{j_x^{k-1}s}(J^{k-1})$$

defined by the formula

$$(d_k^v s)_x(D) = D - (j^{k-1}s \circ \overline{p}_{k-1})_*(D).$$

b) The *structure form of order* k is the 1-form θ^k on J^k with values in the induced vector bundle $V(J^{k-1})_{J^k}$ defined by the formula

$$\theta^k_{(j_x^k s)}(D) = (d_k^v s)_x(\pi_{k,k-1\,*}(D)),$$

where $\pi_{hk} : J^h \to J^k$, for $h \geqslant k$, is the canonical projection.

If $(x_j; y_\alpha^i)_{|\alpha| \leqslant k}$ is the system induced on J^k by a fibred system of local coordinates (x_j, y_i) for the submersion p, locally one has:

$$\theta^k = \sum_i \sum_{|\alpha| < k} \theta_\alpha^i \otimes \frac{\partial}{\partial y_\alpha^i},$$

where θ_α^i is the ordinary 1-form defined by

$$\theta_\alpha^i = dy_\alpha^i - \sum_j y_{\alpha+(j)}^i \, dx_j,$$

and (j) is the multi-index $(j) = (0, \ldots, \underset{(j}{1}, \ldots, 0)$.

c) A vector field D on J^k is called an *infinitesimal contact transformation* if for every linear connection ∇ on $V(J^{k-1})$ there exists an endomorphism f on $V(J^{k-1})_{J^k}$ such that

$$L_D \theta^k = f \circ \theta^k,$$

where L_D is the Lie derivative induced by ∇.

One can prove that for *every* vector field D on Y there is a unique infinitesimal contact transformation $D_{(k)}$ on J^k which is projectable on D. The vector field $D_{(k)}$ is called the infinitesimal contact transformation of order k associated with D.

d) Let J^∞ be the inverse limit of the system (J^k, π_{hk}). The space J^∞ is endowed with a sheaf of rings defined by

$$\mathcal{A}_{J^\infty} = \varinjlim_k \cdot C^\infty_{J^k}.$$

The sections of $\mathcal{A}_{J^\infty}$ are, by definition, the «differentiable functions» on J^∞. Similarly, the differential forms on J^∞ are defined by

$$\Omega^i_{J^\infty} = \varinjlim_k \cdot \Omega^i_{J^k} = \varinjlim_k \cdot \Lambda^i \Omega_{J^k}.$$

For each vector field D on X, there exists a unique vector field $\hat{D}$ on J^∞ projectable on D ans such that

$$\theta^k(\hat{D}) = 0, \quad \text{for every } k \in \mathbb{N}.$$

The vector field $\hat{D}$ is defined by the formula

$$\hat{D}_{(j^\infty_x s)} f = D_x(f \circ j^k s) \quad \text{if } f \in C^\infty(J^k).$$

Locally:

$$\frac{\hat{\partial}}{\partial x_j} = \frac{\partial}{\partial x_j} + \sum_{i,\alpha} y^i_{\alpha+(j)} \frac{\partial}{\partial y^i_\alpha}.$$

One denotes by $\bar{\mathrm{d}} : \Omega^i_{J^\infty} \to \Omega^{i+1}_{J^\infty}$ the *formal differential*. This is the unique anti-derivation of degree $+1$ on the exterior algebra $\oplus_i \Omega^i_{J^\infty}$ such that:

i) $\bar{\mathrm{d}} \circ \mathrm{d} = -\mathrm{d} \circ \bar{\mathrm{d}}$.
ii) $(\bar{\mathrm{d}} f)(\hat{D}) = \hat{D} f$.
iii) $(\bar{\mathrm{d}} f)(D) = 0$, if D is vertical on X.

Locally: $\bar{\mathrm{d}} w = \sum_j \mathrm{d}x_j \wedge L_{\left(\frac{\hat{\partial}}{\partial x_j}\right)} w$.

e) If w, w' are differential forms on a manifold Z with values in the vector bundles E, E^*, respectively, we denote by $w \bar{\wedge} w'$ the exterior product of w and w' with respect to the bi-linear form $E \times_Z E \to Z \times \mathbb{R}$ induced canonically by duality.

Proof of the theorem. Uniqueness. Let $\Phi_{\mathcal{L}}$, $\Phi'_{\mathcal{L}}$ be two $(n+1)$-forms fulfilling the conditions (1), (2) and (3) of the theorem.

From (1) and (2), locally one has

$$\Phi'_{\mathscr{L}} - \Phi_{\mathscr{L}} = \left(\sum_i F_i \, \mathrm{d}y_i\right) \wedge \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n = \left(\sum_i F_i \theta_0^i\right) \wedge \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n,$$

with the notations explained in b). But

$$\eta = \sum_{i,j} \sum_{|\alpha|<r} f^i_{\alpha j} \, \mathrm{d}x_1 \wedge \ldots \wedge \widehat{\mathrm{d}x_j} \wedge \ldots \wedge \mathrm{d}x_n \otimes \mathrm{d}y^i_\alpha$$

and similarly for η'. Therefore, from (3) it follows that:

$$\sum F_i \theta_0^i \wedge \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n = \bar{\mathrm{d}}\left(\sum_{i,j} \sum_{|\alpha|<r} g^i_{\alpha j} \, \mathrm{d}x_1 \wedge \ldots \wedge \widehat{\mathrm{d}x_j} \wedge \ldots \wedge \mathrm{d}x_n \wedge \theta^i_\alpha\right)$$

with $g^i_{\alpha j} = f'^i_{\alpha j} - f^i_{\alpha j}$.

Thus, by writting $D_j = \dfrac{\partial}{\partial x_j}$, one obtains:

$$\sum_i F_i \theta_0^i \wedge \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n =$$

$$= \sum_{i,j} \sum_{|\alpha|<r} (-1)^{j-1} \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n \wedge [(\hat{D}_j g^i_{\alpha j}) \theta^i_\alpha + g^i_{\alpha j} \theta^i_{\alpha+(j)}] =$$

$$= \sum_{i,j} \sum_{|\alpha|<r} (-1)^{j+n-1} [(\hat{D}_j g^i_{\alpha j}) \theta^i_\alpha + g^i_{\alpha j} \theta^i_{\alpha+(j)}] \wedge \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n.$$

So:

(4) $$F_i = \sum_j (-1)^{j+n-1} \hat{D}_j g^i_{0j}.$$

(5) $$\sum_j (-1)^j \hat{D}_j g^i_{\alpha j} + \sum_{\beta+(j)=\alpha} (-1)^j g^i_{\beta j} = 0, \quad \text{for} \quad 0 < |\alpha| < r.$$

(6) $$\sum_{\beta+(j)=\alpha} (-1)^j g^i_{\beta j} = 0, \quad \text{for} \quad |\alpha| = r.$$

Now the function $F_i^k = (-1)^k \sum_j \sum_{|\alpha|=k} (-1)^{j+n-1} \hat{D}^\alpha \hat{D}_j g^i_{\alpha j}$ does not depend on the index $k = 0, \ldots, r-1$.

In fact, from (5) it follows that:

$$F_i^k = (-1)^k \sum_{|\alpha| = k} \hat{D}^{\alpha}\Big(\sum_{j} (-1)^{j+n-1} \hat{D}_j g^i_{\alpha j}\Big) =$$

$$= (-1)^k \sum_{|\alpha| = k} \hat{D}^{\alpha}\Big(\sum_{\beta + (j) = \alpha} (-1)^{j+n} g^i_{\beta j}\Big) =$$

$$= (-1)^{k-1} \sum_{j} \sum_{|\beta| = k-1} (-1)^{j+n-1} \hat{D}^{\beta} \hat{D}_j g^i_{\beta j} = F_i^{k-1}.$$

But $F_i^0 = F_i$ and $F_i^{r+1} = 0$, as it follows from (4) and (6).
Thus, $F_i = 0$; or equivalently $\Phi'_{\mathscr{L}} = \Phi_{\mathscr{L}}$.

EXISTENCE. First let us suppose that p is the canonical projection $p : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$. With the above notations, take

(7)
$$f^i_{\alpha j} = (-1)^{j+n}(1+\alpha_j) \sum_{|\beta| = 0}^{r-1-|\alpha|} (-1)^{|\beta|} \cdot \binom{\alpha + (j) + \beta}{\beta} \frac{|\alpha|!\,|\beta|!}{|\alpha + (j) + \beta|!} \hat{D}^{\beta}\Big(\frac{\partial \mathscr{L}}{\partial y^i_{\alpha + (j) + \beta}}\Big)$$

for the coefficients of the valued form η.

A direct computation proves that

$$\bar{\mathrm{d}}(\eta \overline{\wedge} \theta^r) = \sum_i \sum_{|\alpha| = 1}^{r} (-1)^{|\alpha|} \hat{D}^{\alpha}\Big(\frac{\partial \mathscr{L}}{\partial y^i_{\alpha}}\Big) \theta^i_{\alpha} \wedge \mathrm{d}x_1 \wedge \dots \wedge \mathrm{d}x_n -$$

$$- \sum_i \sum_{|\alpha| = 1}^{r} \frac{\partial \mathscr{L}}{\partial y^i_{\alpha}} \theta^i_{\alpha} \wedge \mathrm{d}x_1 \wedge \dots \wedge \mathrm{d}x_n.$$

Thus, the form

$$\Phi_{\mathscr{L}} = \mathrm{d}\,\mathscr{L} \wedge \omega + \bar{\mathrm{d}}(\eta \wedge \theta^r) = \sum_i \sum_{|\alpha| = 0}^{r} (-1)^{|\alpha|} \hat{D}^{\alpha}\Big(\frac{\partial \mathscr{L}}{\partial y^i_{\alpha}}\Big) \theta^i_0 \wedge \omega$$

verifies conditions (1), (2), (3) and (**) of the theorem.

Finally, let (V_i) be an open cover of the manifold Y such that the submersion $p : V_i \to U_i = p(V_i)$ is isomorphic to the canonical projection $\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, and (ϕ_i) be a partition of unity subordinate to the cover $J^r(V_i/U_i)$. If one denotes by

η_i the valued form associated to the function $\mathscr{L}_i = \phi_i \mathscr{L}$ by formula (7), one has

$$\Phi_{\mathscr{L}_i} = \mathrm{d}\, \mathscr{L}_i \wedge \omega + \overline{\mathrm{d}}(\eta_i \,\overline{\wedge}\, \theta^r).$$

Thus, the form

$$\Phi_{\mathscr{L}} = \sum_i \Phi_{\mathscr{L}_i} = \mathrm{d}\, \mathscr{L} \wedge \omega + \overline{\mathrm{d}}\left(\left(\sum_i \eta_i\right) \overline{\wedge}\, \theta^r\right)$$

fulfils the conditions (1), (2) and (3) of the theorem.

COROLLARY. (Infinitesimal functoriality of the $\Phi_{\mathscr{L}}$ form). *If D is a p-projectable vector field on Y, then:* $L_{D_{(2r)}} \Phi_{\mathscr{L}} = \Phi_{\mathscr{L}'}$, *where* $\mathscr{L}' = D_{(r)} \mathscr{L} + (\operatorname{div} D') \mathscr{L}$ *and D' is the projection of D on X.*

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